On the Size and Complexity of Scrambles

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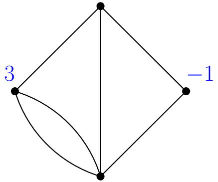
Chip-Firing Games

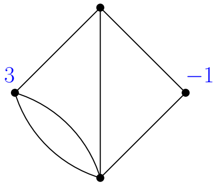
Let G be a connected, loopless multigraph with integers assigned to each vertex



Think of these integers as "chips", with negative integers acting as debts

Chip-Firing Games

In a chip-firing game, we "fire" vertices to move chips around the graph



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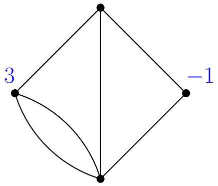

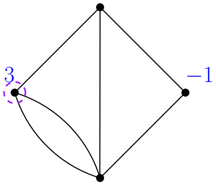

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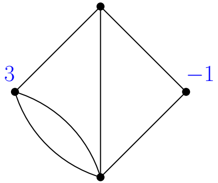

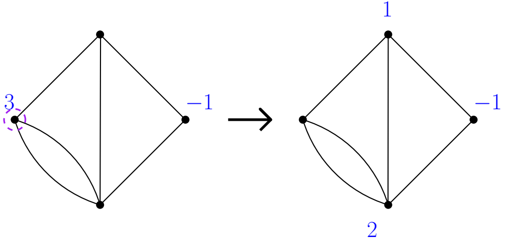

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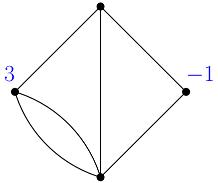

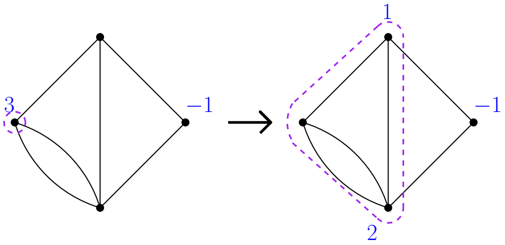

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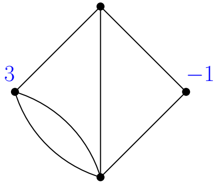
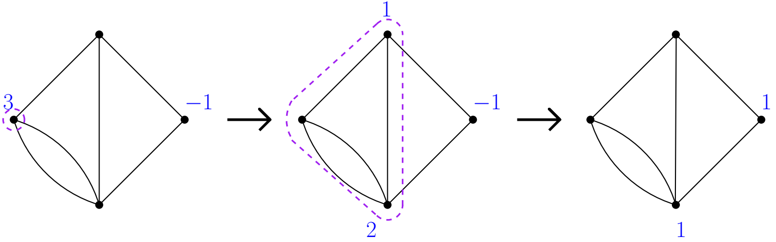


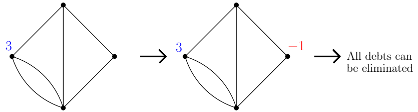


Gonality

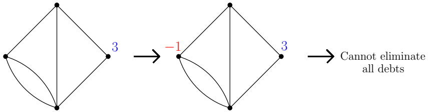
Definition (Gonality)

The **gonality** of a graph G, denoted $\text{gon}(G)$, is the smallest integer k such that there exists a placement of k chips guaranteed to eliminate -1 debt placed anywhere on $V(G)$ without introducing debt elsewhere.

Demo 1:



Demo 2:



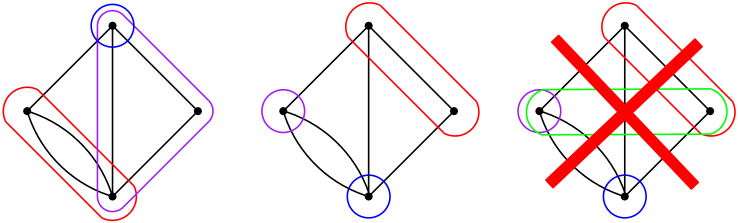
Gonality Facts

1. Graph gonality provides a discrete method for studying the gonality of algebraic curves.
2. Computing gonality is NP-Hard and APX-Hard
3. Given a chip placement, there exists a poly-time algorithm to efficiently determine if it is a winning strategy
4. Harder to lower bound gonality (must show all possible placements of $k-1$ chips are not winning!)

Scrambles

Definition: Scramble

A **scramble** on a graph G, commonly denoted as $\mathcal{S}$, is a collection of connected subgraphs of G (which we call **eggs**).



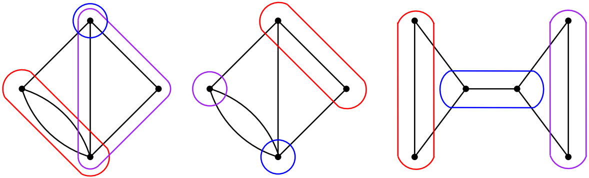
Scrambles

Definition: Hitting Number

The hitting number of a scramble $\mathcal{S}$, denoted $h(\mathcal{S})$, is the minimum number of vertices needed to "hit" each egg of $\mathcal{S}$.

Definition: Egg-Cut Number

The egg-cut number of a scramble $\mathcal{S}$, denoted $e(\mathcal{S})$, is the minimum number of edges needed to be removed to disconnect the graph into two components, each of which completely contains an egg of $\mathcal{S}$.



$h(\mathcal{S}_1) = 2$ $e(\mathcal{S}_1) = 3$

$h(\mathcal{S}_2) = 3$ $e(\mathcal{S}_2) = 3$

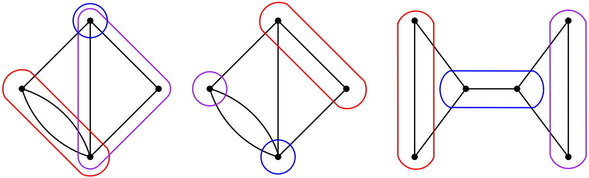
$h(\mathcal{S}_3) = 3$ $e(\mathcal{S}_3) = 1$

Scrambles

Definition: Scramble Order

The **order** of a scramble $\mathcal{S}$, denoted $||\mathcal{S}||$, is:

$$||\mathcal{S}|| = \min\{h(\mathcal{S}), e(\mathcal{S})\}$$

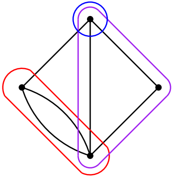


$h(\mathcal{S}_1) = 2$	$h(\mathcal{S}_2) = 3$	$h(\mathcal{S}_3) = 3$
$e(\mathcal{S}_1) = 3$	$e(\mathcal{S}_2) = 3$	$e(\mathcal{S}_3) = 1$
$\vert\vert\mathcal{S}_1\vert\vert = 2$	$\vert\vert\mathcal{S}_2\vert\vert = 3$	$\vert\vert\mathcal{S}_3\vert\vert = 1$

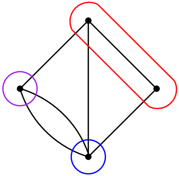
Scrambles

Definition: Scramble Number

The **scramble number** of a graph G, denoted $\text{sn}(G)$, is equal to the maximum possible order of a scramble on G.



$||\mathcal{S}_1|| = 2$



$||\mathcal{S}_2|| = 3$

Scrambles

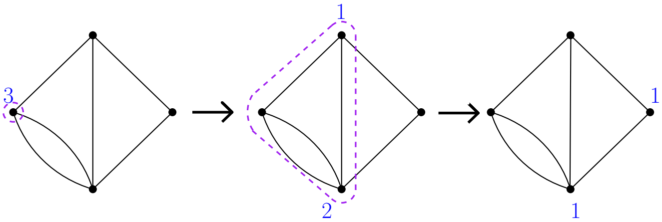
Why we care?

Theorem (Harp et al. 2022)

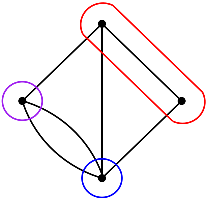
For all graphs G, $\mathrm{sn}(G) \leq \mathrm{gon}(G)$.

Scrambles

$\mathrm{gon}(G) \leq 3$:



$\mathrm{sn}(G) \geq 3$:



$$3 \leq \mathsf{sn}(G) \leq \mathsf{gon}(G) \leq 3 \implies \mathsf{sn}(G) = \mathsf{gon}(G) = 3$$

Scramble Number Facts

1. Scramble number is the best known lower bound on graph gonality
2. Scramble number computation is NP-hard (Echavarria et al. 2021)
3. Unknown whether scramble number is in NP
4. Egg-cut can be computed in poly-time with respect to scramble size

Question

How large do maximum-order scrambles have to be? (In general? For particular graph classes?)

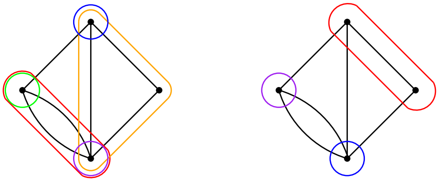
If scrambles are known to be of small size:

1. Possible method of computation/approximation via brute-force
2. Possible NP certificate

Carton Number

Definition (Carton number)

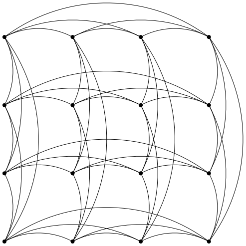
The **carton number** of a graph G, denoted $\text{cart}(G)$, is the minimum size of a maximum order scramble of G. Such a maximum order scramble of minimum size is called a **carton scramble**.



$||\mathcal{S}_1|| = 3$
$|\mathcal{S}_1| = 5$

$||\mathcal{S}_2|| = 3$
$|\mathcal{S}_2| = 3$

Preliminary Counterexample to Vertex Upper Bound



4×4 Rook's Graph on 16 vertices, where two vertices share an edge if and only if they are in the same row or column

Theorem (SMALL 2023)

The 4×4 Rook's Graph has carton number greater than 16.

4×4 Rook's Graph Proof

Theorem (Speeter 2022)

The 4×4 Rook's Graph has scramble number 11.

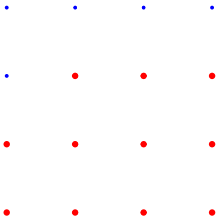
Theorem (SMALL 2023)

Any carton scramble $\mathcal{S}$ satisfies $||\mathcal{S}|| = h(\mathcal{S})$

$\implies$ Any carton scramble of the 4×4 Rook's Graph has hitting number 11

4×4 Rook's Graph Proof

Given any carton scramble $\mathcal{S}$, fix a minimal hitting set A of size 11, and denote $B = V \setminus A$:



4×4 Rook's Graph Proof

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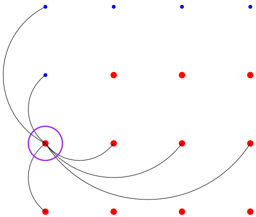

Each vertex has degree 6 $\implies$ no egg of $\mathcal{S}$ contains only a single vertex (or else $||\mathcal{S}||$ would not be as high as 11)

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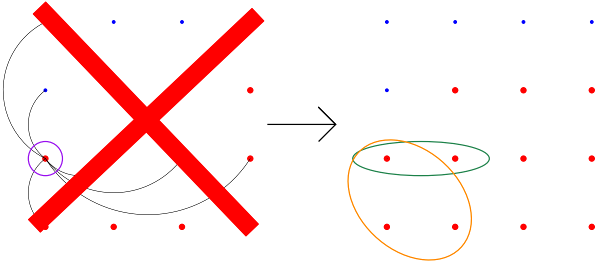


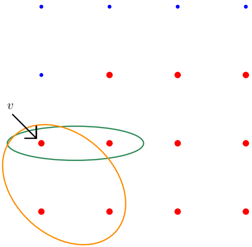
Each vertex has degree 6 $\implies$ no egg of $\mathcal{S}$ contains only a single vertex (or else $||\mathcal{S}||$ would not be as high as 11)

Fact 1

All eggs of $\mathcal{S}$ contain at least 2 vertices.

4 × 4 Rook's Graph Proof

Now, consider any $v \in A$ and the corresponding eggs of $\mathcal{S}$ it hits.



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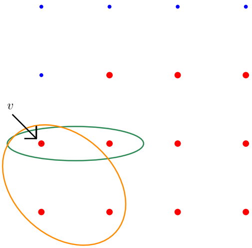

If each egg hit by a vertex $v \in A$ contains another vertex of A, v would not be necessary in the hitting set — contradiction to minimality of A!

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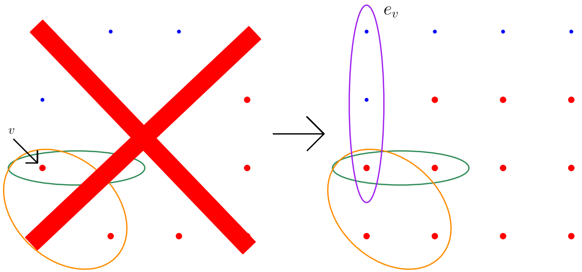

If each egg hit by a vertex $v \in A$ contains another vertex of A, v would not be necessary in the hitting set — contradiction to minimality of A!

Fact 2

Each vertex $v \in A$ must have an an associated egg e_v in $\mathcal{S}$ which does not contain any vertices of $A \backslash \{v\}$.

4×4 Rook's Graph Proof

Fact 1

All eggs of $\mathcal{S}$ contain at least 2 vertices

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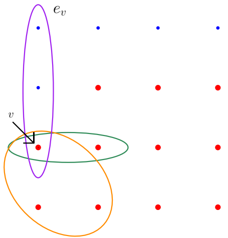
Fact 2

Each vertex $v \in A$ must have an an associated egg e_v in $\mathcal{S}$ which does not contain any vertices of $A \backslash \{v\}$.

Arbitrarily choosing one such e_v for each $v \in A$, each e_v is of size at least 2 but $e_v \cap A = \{v\}$.

Fact 3

Each e_v is hit by at least one vertex of B.



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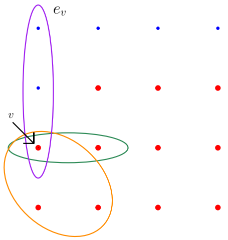

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Each e_v is hit by at least one vertex of B.

We define $\mathcal{S}_B \subseteq \mathcal{S}$ where:

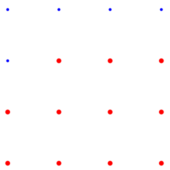
$$\mathcal{S}_B = \{e_v | v \in A\}$$

Note $|\mathcal{S}_B| = 11$ (as $|A| = 11$ and each e_v must be distinct), and B is a hitting set of $\mathcal{S}_B$.

Fact 4

B hits at least 11 eggs of $\mathcal{S}$.

4 × 4 Rook's Graph Proof



Fact 4

B hits at least 11 eggs of $\mathcal{S}$.

Proof by Contradiction

Suppose that our carton scramble $\mathcal{S}$ had at most 16 eggs. Fact 4 $\implies$ there are at most 5 eggs unhit by B!

We can construct a hitting set B' for $\mathcal{S}$ by adding one vertex from each unhit egg to B, and then:

$$|B'| \leq |B| + 5 = 10$$

which implies $||\mathcal{S}|| \leq h(\mathcal{S}) \leq |B'| \leq 10 < 11$ — a contradiction!

Thus, $\mathcal{S}$ must have more than 16 eggs. □

Generalized Lower Bound

Theorem (SMALL 2023)

Let G be a graph on n vertices with maximum degree, $\Delta(G)$, less than $\mathrm{sn}(G)$. Then:

$$\mathrm{cart}(G) \geq 3\,\mathrm{sn}(G) - n$$

NOTE: Letting G be the 4×4 Rook's Graph, $6 = \Delta(G) < \mathrm{sn}(G) = 11$ and the above theorem gives $\mathrm{cart}(G) \geq 17$ as before.

Exponential Lower Bound and Corollaries

Theorem (SMALL 2023)

Let G be a graph on n vertices with $\Delta(G) = d$ and $\text{sn}(G) \geq \lceil c \cdot n^{1/2+\epsilon} \rceil$ for some $c > 1$ and $\epsilon > 0$. Then, $\text{cart}(G) = \Omega(\exp(n^{\epsilon}))$

Proof idea: Consider a scramble $\mathcal{S}$ where $||\mathcal{S}|| \geq \lceil c \cdot n^{1/2+\epsilon} \rceil$

1. We first show that $|A| \geq n^{1/2} \cdot \frac{c}{d+1}$ for all $A \in \mathcal{S}$
2. We then use a probabilistic argument to show the claim

Proof Sketch

Let $\mathcal{S}$ be a scramble of size at least $\lceil c \cdot n^{1/2+\epsilon} \rceil$. Suppose for the sake of contradiction that $\mathcal{S}$ has an egg A with $|A| \leq n^{1/2} \cdot \frac{c}{d+1}$. Then either

Proof Sketch

Let $\mathcal{S}$ be a scramble of size at least $\lceil c \cdot n^{1/2+\epsilon} \rceil$. Suppose for the sake of contradiction that $\mathcal{S}$ has an egg A with $|A| \leq n^{1/2} \cdot \frac{c}{d+1}$. Then either

1. There exists a set $A' \in \mathcal{S}$ such that $A' \cap A = \emptyset$.
   - Then $e(\mathcal{S}) \leq c \cdot n^{1/2}$.

Proof Sketch

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1. There exists a set $A' \in \mathcal{S}$ such that $A' \cap A = \emptyset$.
    - Then $e(\mathcal{S}) \leq c \cdot n^{1/2}$.
2. No such $A' \in \mathcal{S}$ exists.
    - Then A is a hitting set for $\mathcal{S}$. Thus, $h(\mathcal{S}) \leq |A| \leq n^{1/2} \cdot \frac{c}{d+1}$

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2. No such $A' \in \mathcal{S}$ exists.
   - Then A is a hitting set for $\mathcal{S}$. Thus, $h(\mathcal{S}) \leq |A| \leq n^{1/2} \cdot \frac{c}{d+1}$

Both cases lead to a contradiction!

Fact 1

For a scramble $\mathcal{S}$ with $||\mathcal{S}|| \geq \lceil c \cdot n^{1/2+\epsilon} \rceil$, we have that

$$\forall A \in \mathcal{S}. \quad |A| \geq n^{1/2} \cdot \frac{c}{d+1}$$

Proof Sketch

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Let $\ell = \lceil n^{1/2+\epsilon} \rceil$. We construct a hitting set $v_1, ..., v_\ell$ by choosing vertices independently and uniformly at random. We define

Proof Sketch

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Let $\ell = \lceil n^{1/2+\epsilon} \rceil$. We construct a hitting set $v_1, ..., v_\ell$ by choosing vertices independently and uniformly at random. We define

$$X_i^A = \begin{cases} 1 \text{ if } v_i \in A \\ 0 \text{ otherwise} \end{cases}$$

Proof Sketch

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$$X_i^A = \begin{cases} 1 \text{ if } v_i \in A \\ 0 \text{ otherwise} \end{cases}$$

and have

$$\Pr(X_i^A = 1) = \frac{|A|}{|V(G)|} \geq \frac{n^{1/2} \cdot \frac{c}{d+1}}{n} = n^{-1/2} \cdot \frac{c}{d+1}$$

Proof Sketch

Fact 2

For a scramble $\mathcal{S}$ with $||\mathcal{S}|| \geq \lceil c \cdot n^{1/2+\epsilon} \rceil$, we have that

$$\Pr(X_i^A = 1) \geq n^{-1/2} \cdot \frac{c}{d+1}$$

Proof Sketch

Fact 2

For a scramble $\mathcal{S}$ with $||\mathcal{S}|| \geq \lceil c \cdot n^{1/2+\epsilon} \rceil$, we have that

$$\Pr(X_i^A = 1) \geq n^{-1/2} \cdot \frac{c}{d+1}$$

The probability that egg A is unhit is

$$\Pr\left(\sum_{i=1}^{\ell} X_i^A = 0\right) = \Pi_{i=1}^{\ell}\left(1 - \Pr(X_i^A = 1)\right) \leq \left(1 - n^{-1/2} \cdot \frac{c}{d+1}\right)^{\ell}$$
$$\leq \exp\left(-n^{-1/2} \cdot \frac{c}{d+1} \cdot \ell\right) = \exp\left(-n^{\epsilon} \cdot \frac{c}{d+1}\right)$$

Fact 3

For a scramble $\mathcal{S}$ with $||\mathcal{S}|| \geq \lceil c \cdot n^{1/2+\epsilon} \rceil$, we have that

$$\Pr\left(\sum_{i=1}^{\ell} X_i^A = 0\right) \leq \exp\left(-n^{\epsilon} \cdot \frac{c}{d+1}\right)$$

Fact 3

For a scramble $\mathcal{S}$ with $||\mathcal{S}|| \geq \lceil c \cdot n^{1/2+\epsilon} \rceil$, we have that

$$\Pr\left(\sum_{i=1}^{\ell} X_i^A = 0\right) \leq \exp\left(-n^{\epsilon} \cdot \frac{c}{d+1}\right)$$

Lastly,

$$\begin{aligned}
\Pr\left(\{v_1, ..., v_\ell\} \text{ not hitting set for } \mathcal{S}\right) &= \Pr\left(\exists A \in \mathcal{S} : \sum_{i=1}^{\ell} X_i^A = 0\right) \\
&\leq \sum_{A \in \mathcal{S}} \Pr\left(\sum_{i=1}^{\ell} X_i^A = 0\right) \\
&\leq |\mathcal{S}| \cdot \exp\left(-n^{\epsilon} \cdot \frac{c}{d+1}\right)
\end{aligned}$$

Fact 4

For a scramble $\mathcal{S}$ with $||\mathcal{S}|| \geq \lceil c \cdot n^{1/2+\epsilon} \rceil$, we have that

$$\Pr\left(\{v_1, ..., v_\ell\} \text{ not hitting set for } \mathcal{S}\right) \leq |\mathcal{S}| \cdot \exp\left(-n^\epsilon \cdot \frac{c}{d+1}\right)$$

Fact 4

For a scramble $\mathcal{S}$ with $||\mathcal{S}|| \geq \lceil c \cdot n^{1/2+\epsilon} \rceil$, we have that

$$\Pr(\{v_1, ..., v_\ell\} \text{ not hitting set for } \mathcal{S}) \leq |\mathcal{S}| \cdot \exp\left(-n^\epsilon \cdot \frac{c}{d+1}\right)$$

But $\ell \leq ||\mathcal{S}|| \leq h(\mathcal{S})$, so $\{v_1, ..., v_\ell\}$ cannot be a hitting set for $\mathcal{S}$. Therefore, $\Pr(\{v_1, ..., v_\ell\} \text{ not hitting set for } \mathcal{S}) = 1$ and

$$\boxed{|\mathcal{S}| \geq \exp\left(n^\epsilon \cdot \frac{c}{d+1}\right)}$$

□

Consequences

Corollary (SMALL 2023)

The existence of graphs with exponential carton number definitively invalidates scrambles as NP-Certificates in the general case.

Corollary (SMALL 2023)

Minor closed families of graphs with bounded degree have scramble number $O(\sqrt{n})$

Other Results

Theorem (SMALL 2023)

Let G be a graph with minimum degree $\delta(G) \geq \lfloor \frac{n}{2} \rfloor + 1$, $\text{sn}(G)$ and $\text{gon}(G)$ can be 2-approximated in polynomial time.

Theorem (SMALL 2023)

The decision version of disjoint scramble number, where we ask whether $\text{dsn}(G) \geq k$, is fixed parameter tractable when parametrized by k and treewidth.

Theorem (SMALL 2023)

Let G be a planar graph on n vertices with bounded maximum degree. Then, $\text{sn}(G) = O(\sqrt{n})$.

Open Questions

Question 1.1 (SMALL 2023)

Is disjoint scramble number NP-Hard to compute?

Question 1.2 (SMALL 2023)

Is scramble number fixed parameter tractable?

Question 1.3 (SMALL 2023)

Can disjoint scramble number/scramble be approximated to a constant factor in polynomial time?

Thank you!

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Other Results

Theorem (SMALL 2023)

We have $\mathrm{dsn}(G\square H) = \mathrm{sn}(G\square H) = \mathrm{cart}(G\square H) = \mathrm{gon}(G\square H)$ for the following cases.

G	H	Assumptions	$\mathrm{cart}(G\square H)$
T	H	$\frac{\lvert V(H)\rvert}{\lambda(H)} \leq \lvert V(G)\rvert$	$\lvert V(H)\rvert$
P_ℓ	$P_m\square P_n$	$\ell \geq mn/2$	mn
G	$K_\ell\square T$	$\lvert V(G)\rvert \leq \lvert V(T)\rvert, \lvert V(T)\rvert \geq 2$	$\ell\lvert V(G)\rvert$
G	H	$\lambda(H) = \mathrm{gon}(H) = 2, \lvert V(G)\rvert \leq \lvert V(H)\rvert/2$	$2\lvert V(G)\rvert$
G	$K_{m,n}$	$\lvert V(G)\rvert \leq (m+n)/m, m \leq n$	$m\lvert V(G)\rvert$